

Circular Separability

CSI4106 Introduction to Artificial Intelligence

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Data

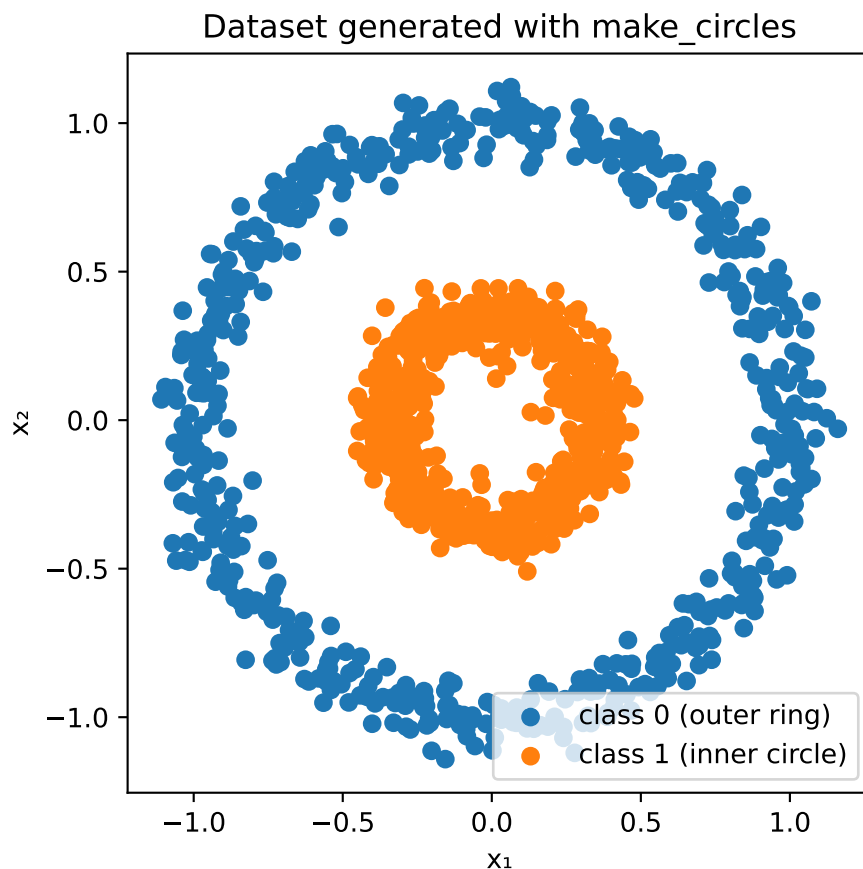
During the lecture, I recommended using [TensorFlow Playground](#) as an interactive tool to deepen your intuitive grasp of neural networks and machine learning principles. In particular, I advised experimenting with the dataset comprising an inner circle (class 1) and an outer ring (class 0). This notebook offers a more detailed exploration of these concepts.

```
# Generate and plot the "circles" dataset
import matplotlib.pyplot as plt
from sklearn.datasets import make_circles

# Generate synthetic data
X, y = make_circles(n_samples=1200, factor=0.35, noise=0.06, random_state=42)

# Separate coordinates for plotting
x1, x2 = X[:, 0], X[:, 1]

# Plot the two classes
plt.figure(figsize=(5, 5))
plt.scatter(x1[y==0], x2[y==0], color="C0", label="class 0 (outer ring)")
plt.scatter(x1[y==1], x2[y==1], color="C1", label="class 1 (inner circle)")
plt.xlabel("x ")
plt.ylabel("x ")
plt.title("Dataset generated with make_circles")
plt.axis("equal") # ensures circles look round
plt.legend()
plt.show()
```



Clearly, this dataset is not linearly separable in (x_1, x_2) !

Feature engineering

When using a linear classifier like [LogisticRegression](#), it is not possible to derive parameters that enable accurate classification of the given examples.

In TensorFlow Playground, users can incorporate two additional features, x_1^2 and x_2^2 . This allows for the classification of examples using a straightforward network configuration with no hidden layers and a single output node. When employing the sigmoid function as the activation, this setup effectively functions as logistic regression. However, the feature space becomes four-dimensional, complicating direct visualization.

In this notebook, we introduce a single feature specifically designed to facilitate visualization.

$$r = x_1^2 + x_2^2,$$

(r) represents the **squared distance from the origin** — essentially the **radius squared** in polar coordinates.

Intuition

- Each point in the original 2-D plane has coordinates (x_1, x_2) .
- If you express those same coordinates in polar form, you have

$$x_1 = r^{1/2} \cos \theta, \quad x_2 = r^{1/2} \sin \theta,$$

or more conventionally, $r_{\text{polar}} = \sqrt{x_1^2 + x_2^2}$.

- Here, we define $r = x_1^2 + x_2^2$, i.e., the **square** of that radius.

Using r instead of $\sqrt{r}$ keeps the mapping differentiable and avoids square roots in the model.

Why it's useful

In the “circle vs. ring” dataset:

- Points from the **inner circle** are close to the origin, small r .
- Points from the **outer ring** are farther away, large r .

Thus, the problem that is **non-linear** in (x_1, x_2) becomes **linearly separable** in r :

$$\text{inner if } r < r^*, \quad \text{outer if } r > r^*.$$

So r is a **feature encoding the radial distance**, allowing a linear model like logistic regression to separate the classes with a single threshold in 1-D.

3D view

(x_1, x_2, r) with $r = x_1^2 + x_2^2$

```
import numpy as np
from sklearn.datasets import make_circles
from sklearn.linear_model import LogisticRegression
import plotly.graph_objects as go

# --- our new feature ---
r = x1**2 + x2**2

# --- fit logistic on r only and get threshold plane ---

clf = LogisticRegression().fit(r.reshape(-1,1), y)
```

```

w = float(clf.coef_[0][0]); b = float(clf.intercept_[0])
r_thresh = -b / w

# --- 3D scatter of (x1, x2, r) ---

scatter = go.Scatter3d(
    x=x1, y=x2, z=r,
    mode="markers",
    marker=dict(size=3, color=y, colorscale="Viridis", showscale=False),
    hovertemplate="x1={x:.3f}<br>x2={y:.3f}<br>r={z:.3f}<extra></extra>",
    name="points"
)

# --- horizontal plane z = r_thresh ---

gx = np.linspace(x1.min()-0.2, x1.max()+0.2, 50)
gy = np.linspace(x2.min()-0.2, x2.max()+0.2, 50)
GX, GY = np.meshgrid(gx, gy)
GZ = np.full_like(GX, r_thresh)

plane = go.Surface(
    x=GX, y=GY, z=GZ,
    opacity=0.35, showscale=False,
    hoverinfo="skip",
    name="p=0.5 plane"
)

fig = go.Figure(data=[plane, scatter])
fig.update_scenes(
    xaxis_title="x ", yaxis_title="x ", zaxis_title="r = x2 + x2",
    aspectmode="cube",
    camera=dict(eye=dict(x=1.6, y=1.6, z=0.9))
)
fig.update_layout(margin=dict(l=0,r=0,b=0,t=20), title=f"Decision plane at r* {r_thresh:.3f}")
fig

## Unfortunatly, plotly graphical objects can only be visualized in HTML, not PDF.
## TODO: Explore https://plotly.com/python/static-image-export/, perhaps this is a workaround

```

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Unable to display output for mime type(s): text/html

Linear separability in 1D (r-space)

Comparing two models, `clf_r` uses a single attribute, $r = x_1^2 + x_2^2$, whereas `quad` includes degree 2 polynomial features.

```
from sklearn.pipeline import Pipeline
from sklearn.preprocessing import PolynomialFeatures
```

```
r = (x1**2 + x2**2).reshape(-1, 1)
```

```
# Models
```

```
clf_r = LogisticRegression().fit(r, y)
```

```
quad = Pipeline([
    ("poly", PolynomialFeatures(degree=2, include_bias=False)),
    ("logreg", LogisticRegression(max_iter=1000))
]).fit(X, y)
```

```
# Threshold  $r^*$  ( $p=0.5$ ) for the r-only model
```

```
w = float(clf_r.coef_[0][0]); b = float(clf_r.intercept_[0])
r_thresh = -b / w if abs(w) > 1e-12 else None
```

```
plt.figure(figsize=(6,4), dpi=140)
```

```
bins = np.linspace(r.min(), r.max(), 40)
```

```
plt.hist(r[y==0], bins=bins, alpha=0.65, label="class 0 (ring)")
```

```
plt.hist(r[y==1], bins=bins, alpha=0.65, label="class 1 (inner)")
```

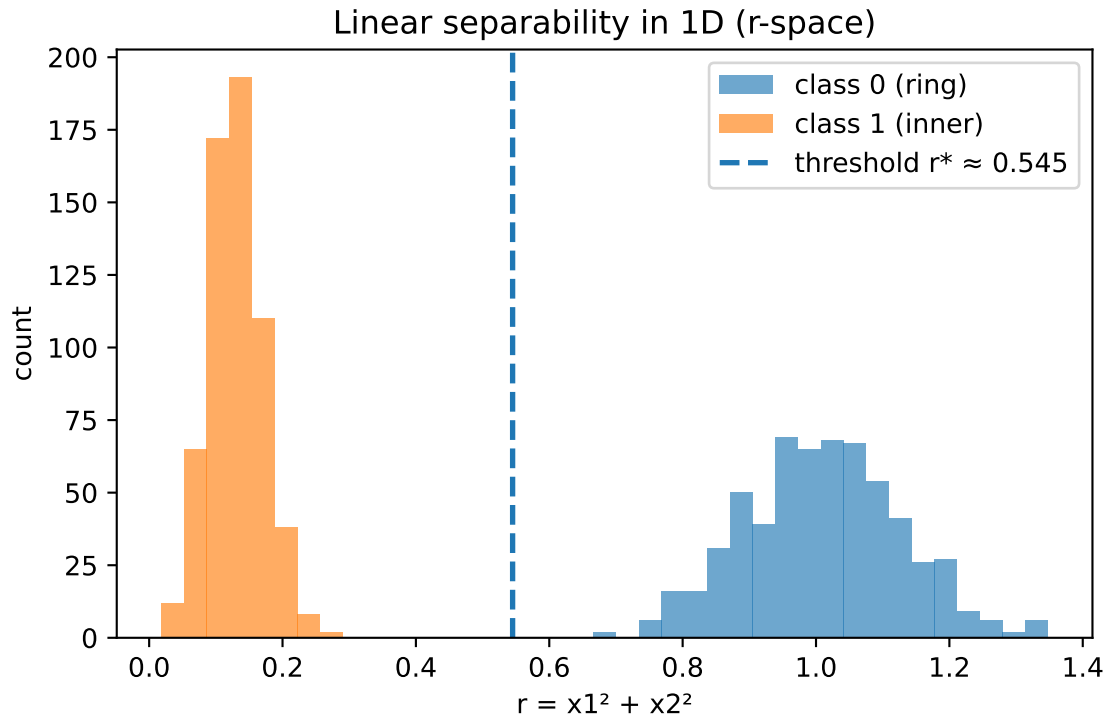
```
if r_thresh is not None:
```

```
    plt.axvline(r_thresh, linestyle="--", linewidth=2, label=f"threshold  $r^*$  {r_thresh:.3f}")
```

```
plt.xlabel("r =  $x_1^2 + x_2^2$ "); plt.ylabel("count")
```

```
plt.title("Linear separability in 1D (r-space)")
```

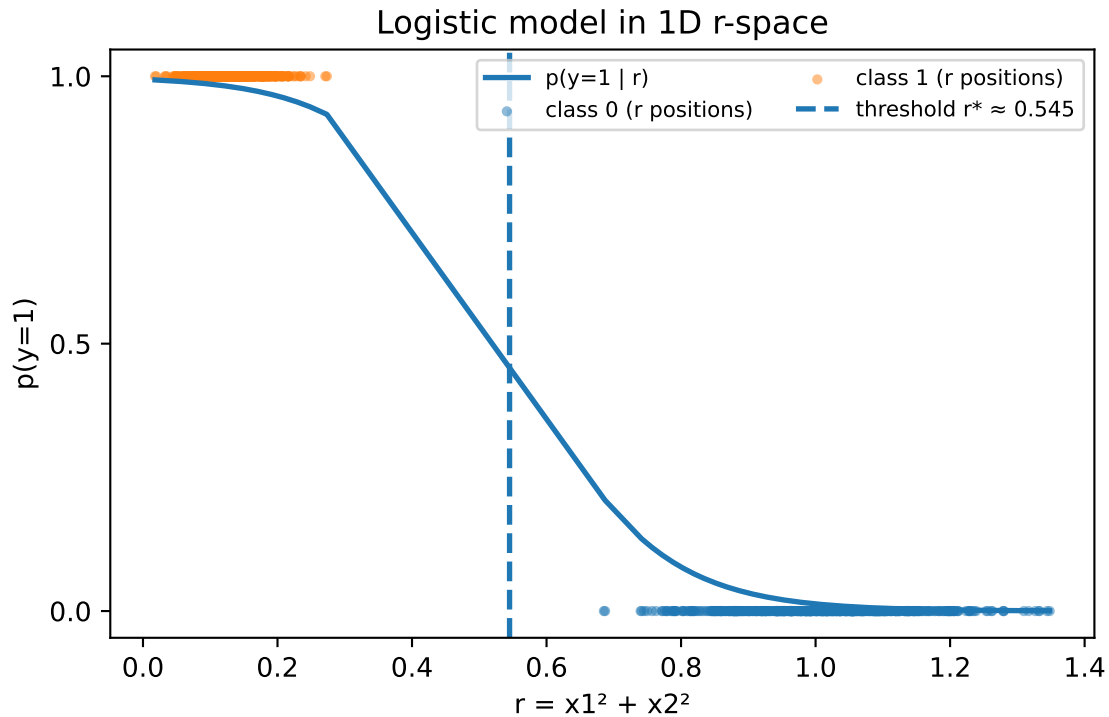
```
plt.legend(); plt.tight_layout(); plt.show()
```



Logistic model in 1D r-space

```
order = np.argsort(r.ravel())
r_sorted = r.ravel()[order]
p_sorted = clf_r.predict_proba(r_sorted.reshape(-1,1))[:,1]

plt.figure(figsize=(6,4), dpi=140)
plt.plot(r_sorted, p_sorted, linewidth=2, label="p(y=1 | r)")
plt.scatter(r[y==0], np.zeros_like(r[y==0]), s=8, alpha=0.5, label="class 0 (r positions)")
plt.scatter(r[y==1], np.ones_like(r[y==1]), s=8, alpha=0.5, label="class 1 (r positions)")
if r_thresh is not None:
    plt.axvline(r_thresh, linestyle="--", linewidth=2, label=f"threshold  $r^* = \{r\_thresh:.3f\}")
plt.yticks([0, 0.5, 1]); plt.ylim(-0.05, 1.05)
plt.xlabel("r =  $x_1^2 + x_2^2$ "); plt.ylabel("p(y=1)")
plt.title("Logistic model in 1D r-space")
plt.legend(loc="best", ncol=2, fontsize=8)
plt.tight_layout(); plt.show()$ 
```



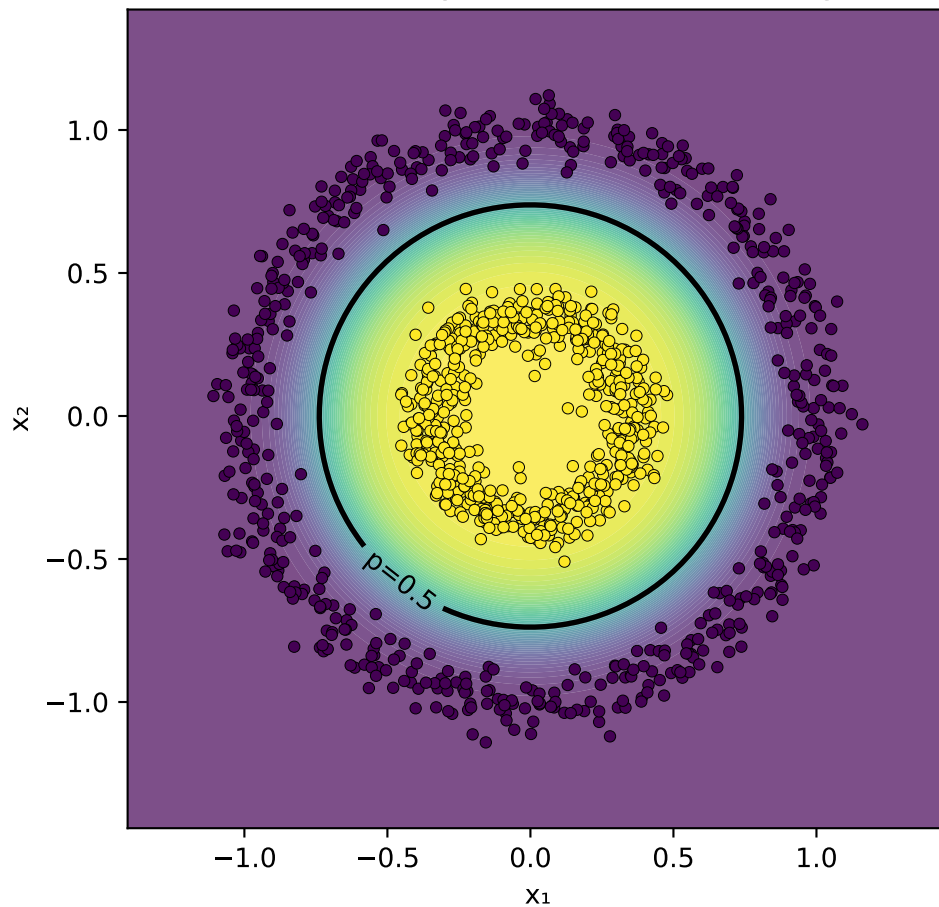
Decision boundary from $r = x_1^2 + x_2^2$ (logistic)

Uses `clf_r.predict_proba` for drawing the decision boundary.

```
xx, yy = np.meshgrid(
    np.linspace(X[:,0].min()-0.3, X[:,0].max()+0.3, 400),
    np.linspace(X[:,1].min()-0.3, X[:,1].max()+0.3, 400)
)
grid = np.c_[xx.ravel(), yy.ravel()]
r_grid = (grid[:,0]**2 + grid[:,1]**2).reshape(-1,1)
p_r = clf_r.predict_proba(r_grid)[:, 1].reshape(xx.shape)

plt.figure(figsize=(5.2,5.2), dpi=140)
cf = plt.contourf(xx, yy, p_r, levels=50, alpha=0.7)
cs = plt.contour(xx, yy, p_r, levels=[0.5], linewidths=2, colors="k")
plt.scatter(X[:,0], X[:,1], c=y, s=18, edgecolor="k", linewidth=0.2)
plt.clabel(cs, fmt="{0.5}:p=0.5")
plt.title("Decision boundary from  $r = x_1^2 + x_2^2$  (logistic)")
plt.xlabel("x "); plt.ylabel("x "); plt.tight_layout(); plt.show()
```

Decision boundary from $r = x_1^2 + x_2^2$ (logistic)

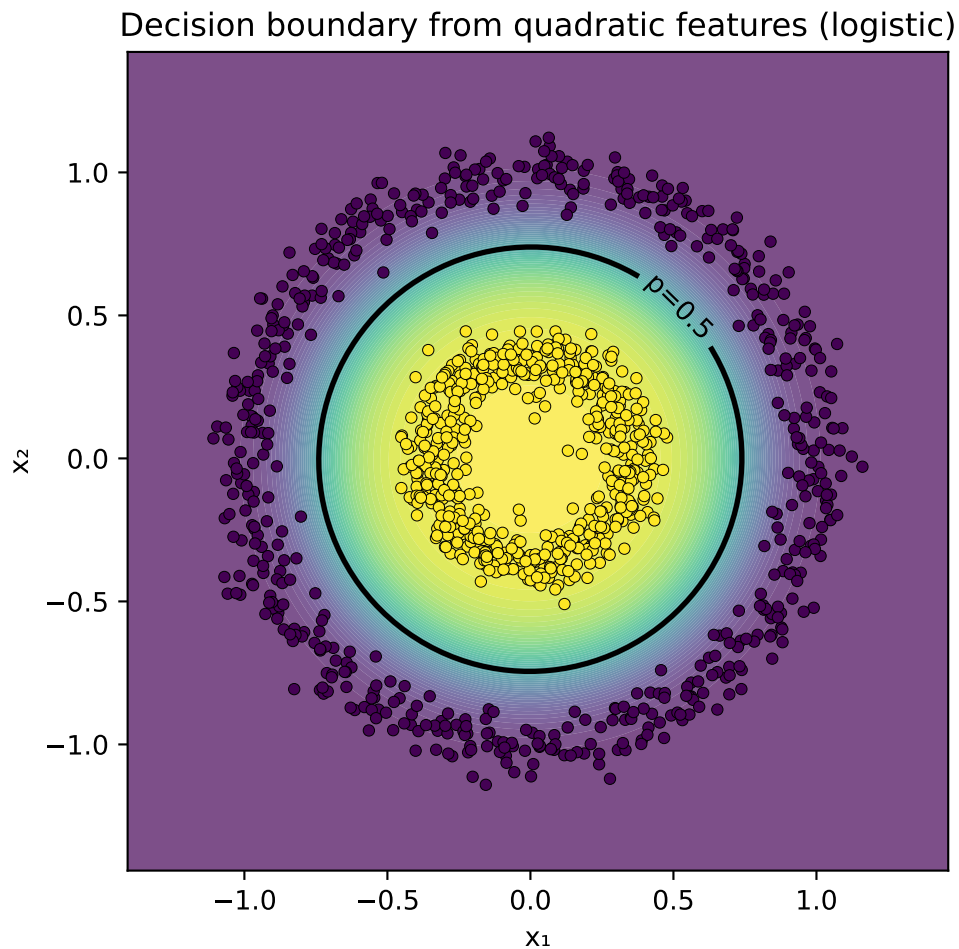


Decision boundary from quadratic features (logistic)

Uses `quad.predict_proba` for drawing the decision boundary. (Indeed, the two models produce the same result.)

```
p_quad = quad.predict_proba(grid)[: , 1].reshape(xx.shape)

plt.figure(figsize=(5.2,5.2), dpi=140)
cf = plt.contourf(xx, yy, p_quad, levels=50, alpha=0.7)
cs2 = plt.contour(xx, yy, p_quad, levels=[0.5], linewidths=2, colors="k")
plt.scatter(X[:,0], X[:,1], c=y, s=18, edgecolor="k", linewidth=0.2)
plt.clabel(cs2, fmt={0.5:"p=0.5"})
plt.title("Decision boundary from quadratic features (logistic)")
plt.xlabel("x "); plt.ylabel("x "); plt.tight_layout(); plt.show()
```



Neural network

To further our understanding of neural networks behavior, let's configure a model to use only the features x_1 and x_2 . We will set up two hidden layers, each consisting of four neurons. In my experiments, the **Tanh** activation function led to rapid network convergence. The **Sigmoid** function also converged, though at a slower pace. Meanwhile, the **ReLU** activation function enabled quick convergence but produced a decision boundary comprised of linear segments.

```
from sklearn.model_selection import train_test_split

# Two-layer (4 + 4) Keras MLP on make_circles + decision boundary plot
```

```
import tensorflow as tf
from tensorflow import keras
from tensorflow.keras import layers
```

```
# 0) Reproducibility
```

```
np.random.seed(42)
tf.random.set_seed(42)
```

```
# 1) Data
```

```
X_train, X_valid, y_train, y_valid = train_test_split(
    X, y, test_size=0.25, random_state=42, stratify=y
)
```

```
# 2) Model: 2 hidden layers with 4 units each
# tanh works nicely here due to circular symmetry (ReLU is fine too).
```

```
model = keras.Sequential([
    layers.Input(shape=(2,)),
    layers.Dense(4, activation="tanh"),
    layers.Dense(4, activation="tanh"),
    layers.Dense(1, activation="sigmoid")
])
```

```
model.compile(optimizer=keras.optimizers.Adam(learning_rate=0.01),
              loss="binary_crossentropy",
              metrics=["accuracy"])
```

```
# Optional: early stopping for quick convergence
```

```
es = keras.callbacks.EarlyStopping(patience=20, restore_best_weights=True, monitor="val_loss")
```

```
history = model.fit(
    X_train, y_train,
    validation_data=(X_valid, y_valid),
    epochs=300,
    batch_size=64,
    verbose=0,
    callbacks=[es]
)
```

```
print(f"Train acc: {history.history['accuracy'][-1]:.3f} | "
      f"Val acc (best): {max(history.history['val_accuracy']):.3f}")
```

Train acc: 1.000 | Val acc (best): 1.000

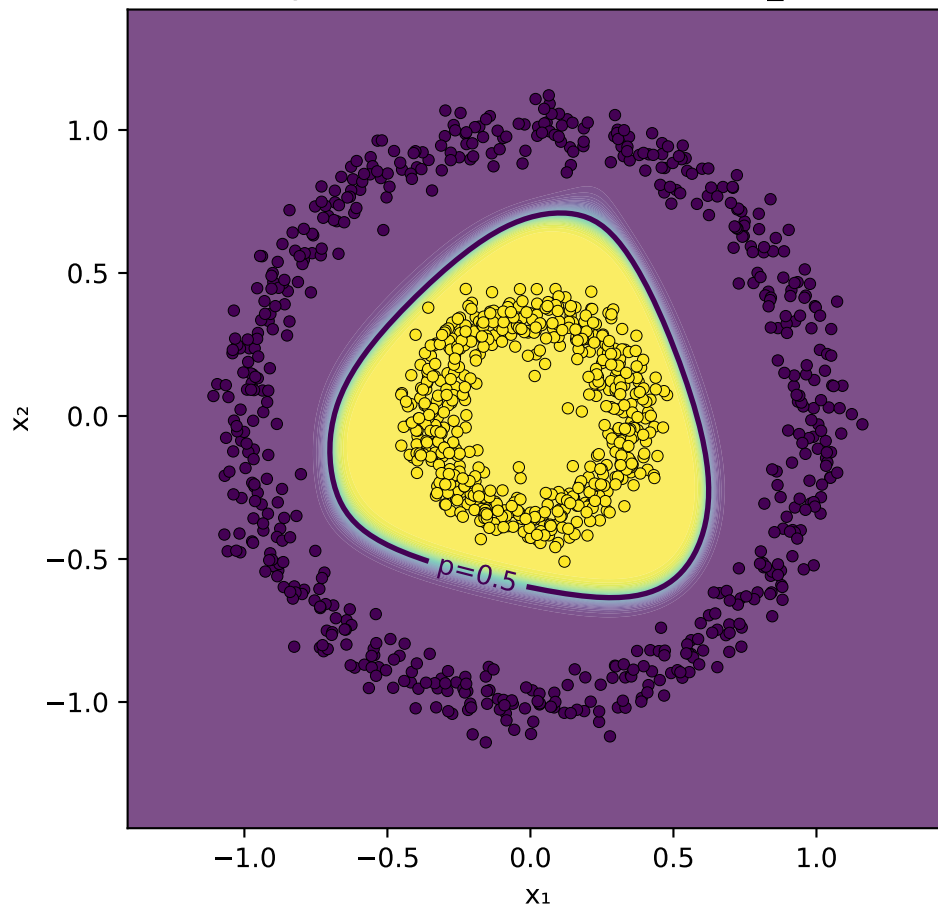
3) Plot helper: decision boundary in the original (x1, x2) plane

```
def plot_decision_boundary(model, X, y, title="Keras MLP decision boundary"):
    # grid over the input plane
    pad = 0.3
    x1_min, x1_max = X[:,0].min()-pad, X[:,0].max()+pad
    x2_min, x2_max = X[:,1].min()-pad, X[:,1].max()+pad
    xx, yy = np.meshgrid(
        np.linspace(x1_min, x1_max, 400),
        np.linspace(x2_min, x2_max, 400)
    )
    grid = np.c_[xx.ravel(), yy.ravel()]
    # predict probabilities on the grid
    p = model.predict(grid, verbose=0).reshape(xx.shape)

    # filled probabilities + p=0.5 contour + data points
    plt.figure(figsize=(5.2, 5.2), dpi=140)
    plt.contourf(xx, yy, p, levels=50, alpha=0.7)
    cs = plt.contour(xx, yy, p, levels=[0.5], linewidths=2)
    plt.scatter(X[:,0], X[:,1], c=y, s=18, edgecolor="k", linewidth=0.2)
    plt.clabel(cs, fmt={0.5: "p=0.5"})
    plt.title(title)
    plt.xlabel("x ")
    plt.ylabel("x ")
    plt.tight_layout()
    plt.show()
```

```
plot_decision_boundary(model, X, y, title="Two-layer (4+4) Keras MLP on make_circles")
```

Two-layer (4+4) Keras MLP on make_circles



XOR dataset

This example uses a classical dataset, characterized by an “exclusive-or” (XOR) pattern, which is frequently employed to demonstrate non-linear decision boundaries and to underscore the necessity of neural networks in addressing such complexities.

```
# 1. Generate the XOR dataset

n_samples = 800
rng = np.random.default_rng(42)

# Features uniformly sampled in [-6, 6]
X = rng.uniform(-6, 6, size=(n_samples, 2))
x1, x2 = X[:, 0], X[:, 1]
```

```

# Labels: positive if (x1 and x2) have the same sign (top-left or bottom-right)
y = ((x1 * x2) > 0).astype(int)

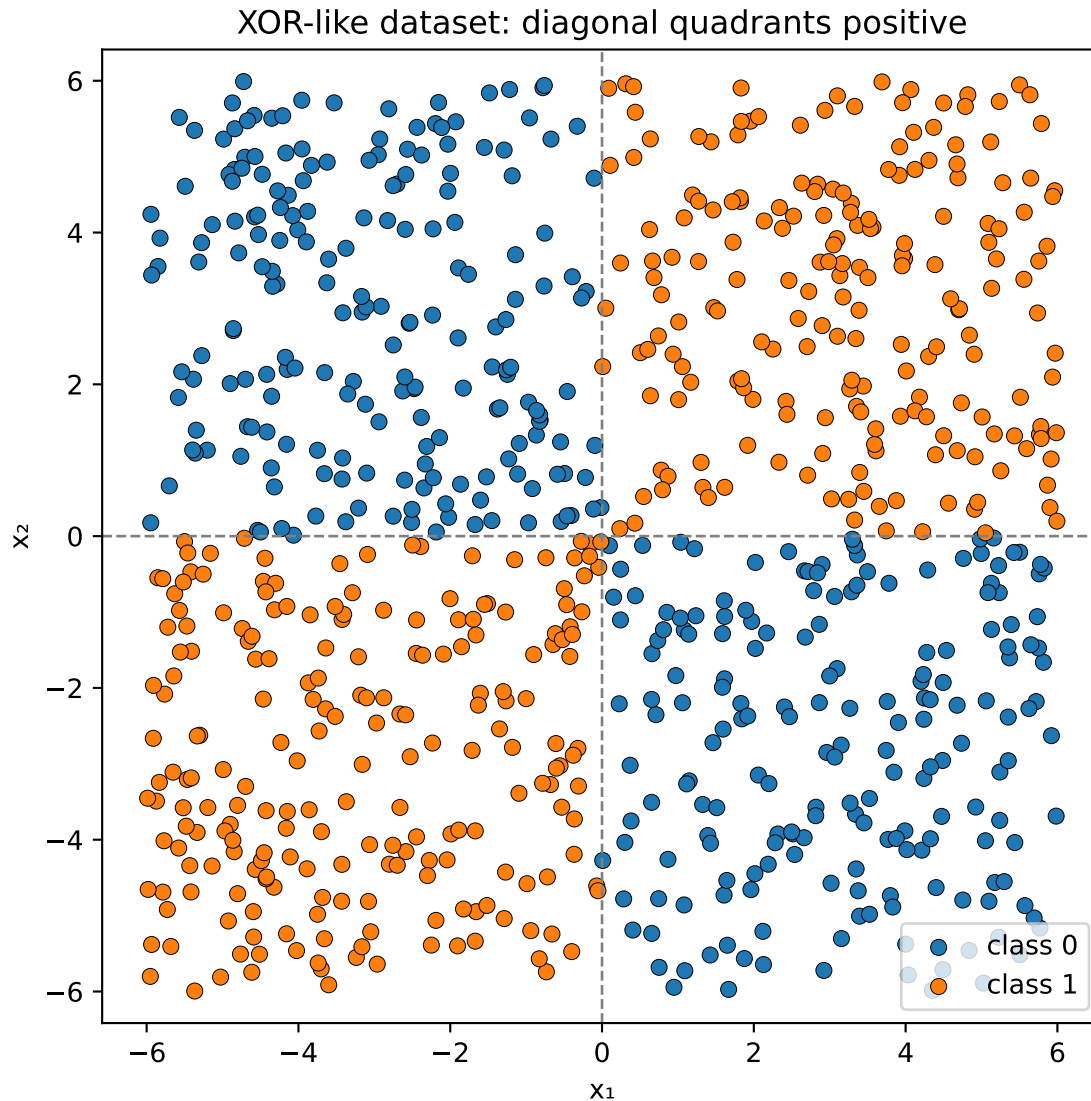
# 2. Visualize the data

plt.figure(figsize=(6, 6))
plt.scatter(X[y == 0, 0], X[y == 0, 1],
            color="C0", label="class 0", edgecolor="k", linewidth=0.3)
plt.scatter(X[y == 1, 0], X[y == 1, 1],
            color="C1", label="class 1", edgecolor="k", linewidth=0.3)

# Draw axes for clarity
plt.axhline(0, color="gray", linestyle="--", linewidth=1)
plt.axvline(0, color="gray", linestyle="--", linewidth=1)

plt.xlabel("x ")
plt.ylabel("x ")
plt.title("XOR-like dataset: diagonal quadrants positive")
plt.xlim(-6, 6)
plt.ylim(-6, 6)
plt.axis("equal")
plt.legend()
plt.tight_layout()
plt.show()

```



Would it be reasonable to anticipate that the features engineered for circular data could be beneficial in this context?

```
r = (x1**2 + x2**2).reshape(-1, 1)

# Models
clf_r = LogisticRegression().fit(r, y)

quad = Pipeline([
    ("poly", PolynomialFeatures(degree=2, include_bias=False)),
    ("logreg", LogisticRegression(max_iter=1000))
])
```

```

]).fit(X, y)

# Threshold  $r^*$  ( $p=0.5$ ) for the  $r$ -only model

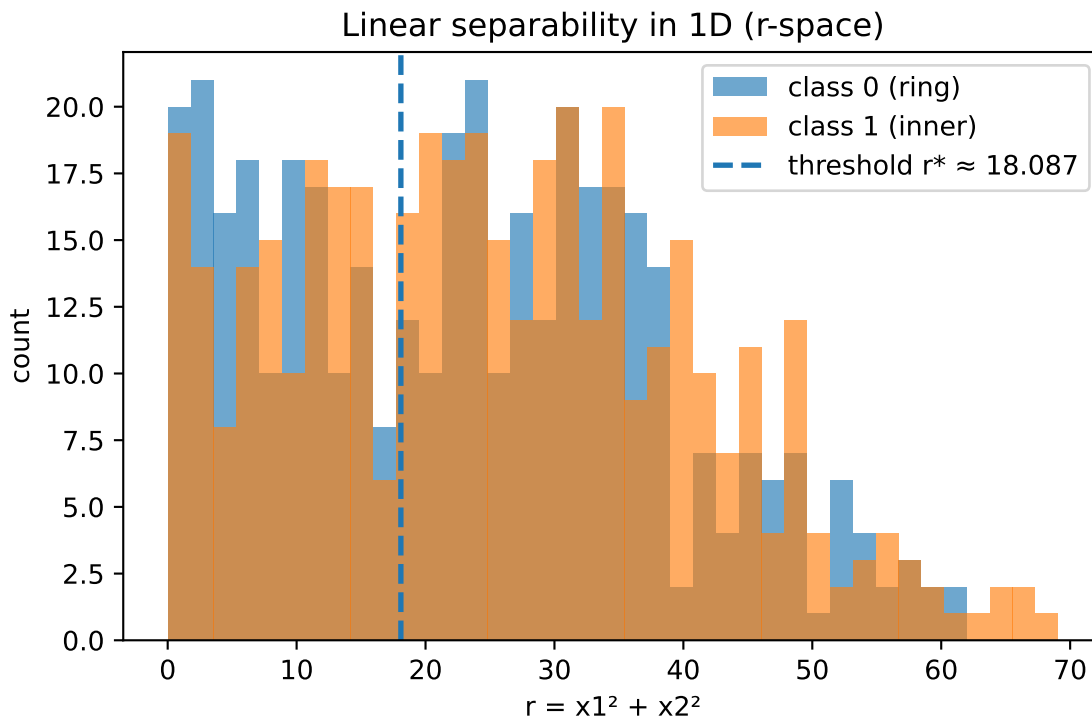
w = float(clf_r.coef_[0][0]); b = float(clf_r.intercept_[0])
r_thresh = -b / w if abs(w) > 1e-12 else None

```

Certainly not!"

```

plt.figure(figsize=(6,4), dpi=140)
bins = np.linspace(r.min(), r.max(), 40)
plt.hist(r[y==0], bins=bins, alpha=0.65, label="class 0 (ring)")
plt.hist(r[y==1], bins=bins, alpha=0.65, label="class 1 (inner)")
if r_thresh is not None:
    plt.axvline(r_thresh, linestyle="--", linewidth=2, label=f"threshold  $r^* = \{r\_thresh:.3f\}")
plt.xlabel("r =  $x_1^2 + x_2^2$ "); plt.ylabel("count")
plt.title("Linear separability in 1D (r-space)")
plt.legend(); plt.tight_layout(); plt.show()$ 
```



To address the problem using logistic regression, it would be necessary to engineer new features. Can you propose any such features?

What about our neural network model? Can we reasonably expect it to perform well on this dataset? Below is a verbatim copy of the code.

```
# 0) Reproducibility
```

```
np.random.seed(42)
tf.random.set_seed(42)
```

```
# 1) Data
```

```
X_train, X_valid, y_train, y_valid = train_test_split(
    X, y, test_size=0.25, random_state=42, stratify=y
)
```

```
# 2) Model: 2 hidden layers with 4 units each
# tanh works nicely here due to circular symmetry (ReLU is fine too).
```

```
model = keras.Sequential([
    layers.Input(shape=(2,)),
    layers.Dense(4, activation="tanh"),
    layers.Dense(4, activation="tanh"),
    layers.Dense(1, activation="sigmoid")
])
```

```
model.compile(optimizer=keras.optimizers.Adam(learning_rate=0.01),
              loss="binary_crossentropy",
              metrics=["accuracy"])
```

```
# Optional: early stopping for quick convergence
```

```
es = keras.callbacks.EarlyStopping(patience=20, restore_best_weights=True, monitor="val_loss")
```

```
history = model.fit(
    X_train, y_train,
    validation_data=(X_valid, y_valid),
    epochs=300,
    batch_size=64,
    verbose=0,
    callbacks=[es]
)
```

```
print(f"Train acc: {history.history['accuracy'][-1]:.3f} | "
      f"Val acc (best): {max(history.history['val_accuracy']):.3f}")
```

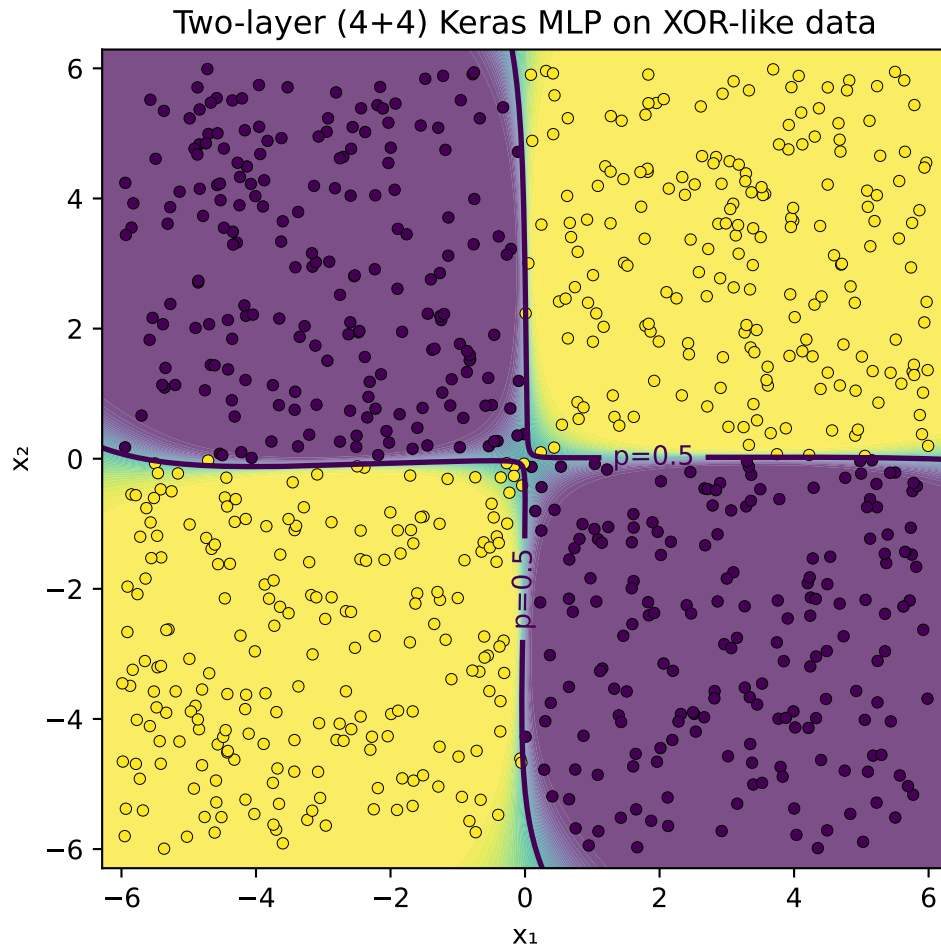
Train acc: 0.992 | Val acc (best): 0.995

3) Plot helper: decision boundary in the original (x1, x2) plane

```
def plot_decision_boundary(model, X, y, title="Keras MLP decision boundary"):
    # grid over the input plane
    pad = 0.3
    x1_min, x1_max = X[:,0].min()-pad, X[:,0].max()+pad
    x2_min, x2_max = X[:,1].min()-pad, X[:,1].max()+pad
    xx, yy = np.meshgrid(
        np.linspace(x1_min, x1_max, 400),
        np.linspace(x2_min, x2_max, 400)
    )
    grid = np.c_[xx.ravel(), yy.ravel()]
    # predict probabilities on the grid
    p = model.predict(grid, verbose=0).reshape(xx.shape)

    # filled probabilities + p=0.5 contour + data points
    plt.figure(figsize=(5.2, 5.2), dpi=140)
    plt.contourf(xx, yy, p, levels=50, alpha=0.7)
    cs = plt.contour(xx, yy, p, levels=[0.5], linewidths=2)
    plt.scatter(X[:,0], X[:,1], c=y, s=18, edgecolor="k", linewidth=0.2)
    plt.clabel(cs, fmt={0.5: "p=0.5"})
    plt.title(title)
    plt.xlabel("x ")
    plt.ylabel("x ")
    plt.tight_layout()
    plt.show()
```

```
plot_decision_boundary(model, X, y, title="Two-layer (4+4) Keras MLP on XOR-like data")
```



Pretty impressive, don't you think?

The key takeaway is that feature engineering combined with basic machine learning models can yield satisfactory results, especially in straightforward scenarios like those illustrated here. When data visualization is feasible or when domain expertise is available, this process is generally accessible. However, in more complex situations involving hundreds or thousands of features and intricate domains, neural networks are particularly effective. Their capability to learn hierarchical representations of features allows them to excel in these challenging contexts.